(6) MATHEMATICAL STUDY OF BACKGROUND "NOISE"

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INTRODUCTION AND SUMMARY

In designing guidance or detection systems, an important aid towards optimal or quasi-optimal design is a knowledge of the properties of the background in which the system must work. The present paper is concerned with the effect of background gradients on infrared systems that use a rotating reticle in the focal plane of a scanning system as an aid in discriminating against background signals in favor of desired or target signals.

The background radiation is regarded here as a two-dimensional random function of space or angular variables. (The idea of treating the background in this manner is not original with the writer but has appeared in the works of R. C. Jones and perhaps others. The writer knows of much important work by Dr. Jones in this field.)

The scanning and rotating action may be said to convert the background or <u>input</u> random process or "noise" into a new noise process at the output side of the reticle. If this output noise can be described in terms of an output correlation function or power spectrum, it becomes possible, in a limited sense, to calculate the effect of processing by photoconductive cells, amplifiers and the like. Thus, the system analysis stage of design may be facilitated and possibly a little progress made toward more sophisticated solutions of the design synthesis problem.

In this paper, expressions are derived for the output autocorrelation quantity $\langle y(t)\, y(t+\tau) \rangle$ where $y(t)$ and $y(t+\tau)$ are the reticle outputs at time t and $t+\tau$ respectively, and $\langle\ \rangle$ refers to an ensemble average. If the background random function may be regarded as a stationary random function, this autocorrelation function may be converted by Fourier transformation to an output power spectrum.

The analysis, which is thought to contain some new results, has the advantage of considerable generality and, at the same time, has sufficient flexibility to allow computation in simpler cases. The work serves also to clarify the roles of the pertinent variables in the over-all problem. If fullest generality is maintained, the expressions obtained

can be converted into numbers only with the aid of very extensive computational aid. In certain cases, however, the expressions may be reduced to relatively illuminating forms, and the computation required may be reduced to a moderate amount.

The form of some of the results obtained serves to point up the type of background measurements that would be necessary for more accurate estimates of the outputs. The autocorrelation function of the background enters, in a fundamental way, into the expressions; it may therefore be advisable to design the measurement program to measure directly this correlation function. For some background "bucking" or "suppression" schemes, success in cancelling out background depends ultimately upon the existence of relations or, somewhat less generally, of cross-correlations between the random processes in the various "colors." This would suggest the investigation of a program of measuring these cross-correlations.

A separate study (Appendix B) is made of the problem of studying background by a technique of circular scanning; that is, scanning the background with a small field of view that moves repeatedly around a circle, and relating the Fourier coefficients of the resulting periodic function to the power spectral or correlation properties of the background. An analysis is made of the processes required to effect this conversion of the measurement data. The limitations of this procedure are such as to emphasize the desirability of having direct measurements of the correlation function.

BASIC FORMULATION

Consider an optical system with a rotating reticle of radius a, in the focal plane, scanning at time t a portion of the background in a certain region of the spectrum e.g., the neighborhood of 2 microns. Use a set of polar coordinates (r, θ) (Figure 1) to describe position on the reticle. Since that portion of the background in the instantaneous field of view is imaged in the reticle plane, we may also consider r, θ to be the angular polar coordinates of a point in the background. Let the transmission function of the reticle be $R(r, \theta)$, and the radiance function of the background to be $B(r, \theta)$. Then the expression

$$y(t) = \int_0^a dr \int_0^{2\pi} d\theta \, r \, B(r, \theta) \, R(r, \theta) \ , \tag{1}$$

may be regarded as proportional to the output of the reticle at time t.[1] The factors of proportionality omitted involve the size of the collecting system and the transmission factor of the optical system. Henceforth we shall refer to y(t) simply as the "output."

Let the scanning system move with velocity v in the direction of the reference angle $\theta = 0$ (Figure 1) so that at time $t + \tau$ the center of the field of view has moved an angular distance $v\tau$ in the direction of $\theta = 0$. Let the coordinate system translate with this scanning motion and denote the coordinate system for the translated reticle by (r', θ'). The output at time $t + \tau$ is

$$y(t + \tau) = \int_0^a dr' \int_0^{2\pi} d\theta' \, r' \, B(r', \theta') \, R(r', \theta' - \omega\tau) \ , \tag{2}$$

[1] For this to be strictly true $B(r, \theta)$ should not be the true radiance function, but the radiance function as modified by the aberration of the system. The analysis applies, strictly speaking, to this modified definition of $B(r, \theta)$. If the aberration is small the two quantities are almost the same.

Another modification is necessary if the image of the field of view does not coincide with the reticle circle, or if there is another significant transmission function of (r, θ) in addition to that supplied by the reticle. For such a case, the work through Equation (4) would be unchanged except that $B(r, \theta)$ would be multiplied by an "aperture function" say $A(r, \theta)$.

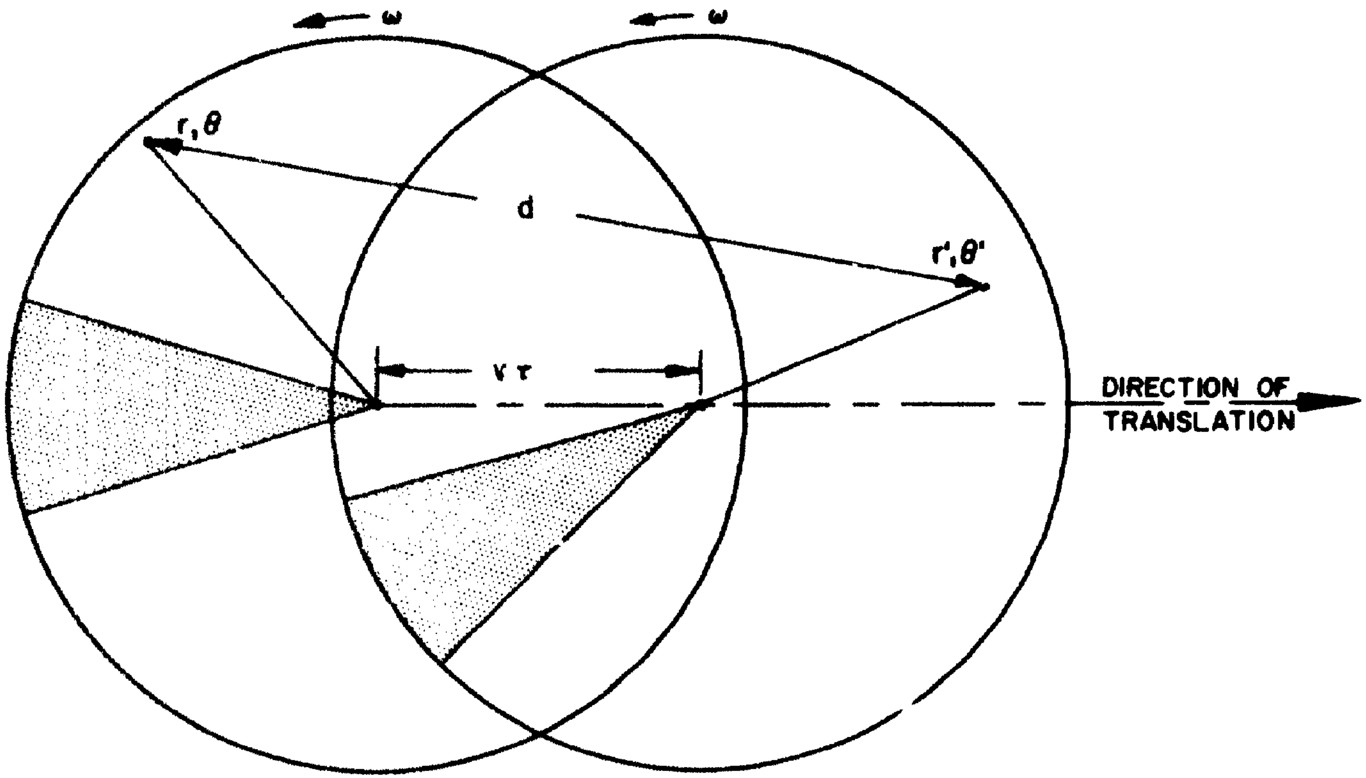

FIG. 1 ROTATING AND TRANSLATING SYSTEM.
THE SHADED SECTION INDICATES A RUDIMENTARY RETICLE FUNCTION.

since the reticle, which is rotating with angular velocity ω, has rotated through the angle $\omega\tau$ while translating through the distance $v\tau$.

Let us form the product

$$y(t)y(t+\tau) = \int_0^a dr \int_0^a dr' \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\, rr'\, R(r,\theta)R(r',\theta'-\omega\tau)B(r,\theta)B(r',\theta') \tag{3}$$

Now the ensemble average of $y(t)\,y(t+\tau)$, that is the value of $y(t)\,y(t+\tau)$ averaged over various samples of the background, is

$$\langle y(t)y(t+\tau)\rangle = \int_0^a dr \int_0^a dr' \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\, rr'\, R(r,\theta)\,R(r',\theta'-\omega\tau)\langle B(r,\theta)\,B(r',\theta')\rangle \tag{4}$$

where the brackets $\langle\ \rangle$ mean ensemble average. The quantity $\langle y(t)y(t+\tau)\rangle$ is the autocorrelation function of the reticle output $y(t)$. If the background random function $B(r,\theta)$ may be regarded as a "stationary" random function that is, roughly speaking, if its average properties do not vary with[2] (r,θ) over a region interest, then we can define a useful "power spectral density" or "power spectrum" which is the Fourier transform of $\langle y(t)y(t+\tau)\rangle$. The effect of subsequent processing upon the reticle output may then be easily computed in a limited statistical sense.

The central goal of our analysis is then to evaluate $\langle y(t)y(t+\tau)\rangle$ from Equation (4). The background quantity that we need to compute this is $\langle B(r,\theta)\,B(r',\theta')\rangle$ which is the correlation function of the background. Now, in particular cases, even if $B(r,\theta)$ may be regarded as a stationary function, it may happen that $\langle B(r,\theta)\,B(r',\theta')\rangle$ must be regarded as a function of the vector distance between the points (r,θ) and (r',θ'). If this

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[2] For a more detailed mathematical description of a stationary random process the reader is referred to the extensive mathematical literature in the subject of random functions.

is true then the evaluation of Equation (4) will probably be extremely difficult (but still possible with the aid of the high-speed digital computers). This case will not be considered further here.

ADDITIONAL ASSUMPTIONS

In order to make the general result of Equation (4) more amenable to further analysis, we shall make some simplifying assumptions. An assumption that appears to be useful, is that the autocorrelation function $\langle B(r,\theta)\, B(r',\theta')\rangle$ is a function $\psi(d)$ only of the scalar distance d between (r,θ) and (r',θ'); in this case we speak of the background random function as being isotropic. With the aid of Figure 1 it is seen that the distance d is given by

$$\begin{aligned} d^2 &= (r'\cos\theta + v\tau - r\cos\theta)^2 + (r'\sin\theta' - r\sin\theta)^2 \\ &= r'^2 + r^2 - 2rr'\cos(\theta' - \theta) + 2v\tau(r'\cos\theta' - r\cos\theta) + (v\tau)^2 . \end{aligned} \tag{5}$$

With this assumption, Equation (4) takes the form

$$\langle y(t)y(t+\tau)\rangle = \int_0^a \int_0^a \int_0^{2\pi} \int_0^{2\pi} dr\, dr'\, d\theta\, d\theta'\, rr'\, R(r,\theta)\, R(r',\theta' - \omega\tau)\, \psi(d) . \tag{4a}$$

Another simplifying, but useful, assumption is that $R(r,\theta)$ is a function only of θ. There are cases when reasonably simple results can be obtained for more general functions $R(r,\theta)$, but our illustrative examples will be limited to the simpler case. Another special assumption of useful scope is that the scanning speed v is small compared with the speed of most points of the rotating reticle, so that the error introduced in setting $v = 0$ in Equation (5) is not serious.[3] The important simplification resulting from letting $v = 0$ is that d becomes a function of $\theta' - \theta$.

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[3] Incidentally, for the case in which the field of view is rectangular and the reticle motion consists of parallel bars sweeping across the field of view, the assumption $v = 0$ is not necessary; i.e., results similar to those of the subsequent example may be obtained even if $v \neq 0$.

A CASE OF INTEREST

With these assumptions and the notation $Q(\tau) \equiv \langle y(t)\, y(t + \tau)\rangle$, Equation (4) becomes

$$Q(\tau) = \int_0^a dr \int_0^a dr' \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\, r r'\, R(\theta)\, R(\theta' - \omega\tau)\, \psi(d) , \tag{6}$$

where

$$d = \left[r^2 + r'^2 - 2rr' \cos(\theta' - \theta)\right]^{1/2} . \tag{7}$$

The expression (6) has some interesting properties. First, let us rewrite it as

$$Q(\tau) = \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\, G(\theta' - \theta)\, R(\theta)\, R(\theta' - \omega\tau) , \tag{8}$$

where

$$G(\theta' - \theta) = \int_0^a dr \int_0^a dr'\, rr' \psi\left[r'^2 + r^2 - 2rr' \cos(\theta' - \theta)\right]^{1/2} . \tag{9}$$

It will be convenient to write u for $\theta' - \theta$, giving

$$G(u) = \int_0^a dr \int_0^a dr'\, rr' \psi\left[r'^2 + r^2 - 2rr' \cos u\right]^{1/2} . \tag{9a}$$

An important result relative to Equation (8) may now be stated: $Q(\tau)$ may be written as

$$Q(\tau) = \int_0^{2\pi} du\, G(u + \omega\tau)\, P(u) , \tag{10}$$

where

$$P(u) = \int_0^{2\pi} d\theta\, R(\theta)\, R(\theta + u) . \tag{11}$$

The method by which this result was obtained is given in detail in Appendix A. The "intuitive" significance of Equation (10) and (11) is as follows: The form of Equation (11) indicates that $P(u)$ is a type of correlation function to be called hereafter the "reticle correlation function." $Q(\tau)$ is the correlation function of the output as previously discussed. Equation (10) expresses $Q(\tau)$ as a convolution of $G(u)$ and $P(u)$. We may therefore regard $G(u)$ as another correlation function. It is the correlation function of the background as "seen" by the rotating reticle; it may be called the "noise correlation function."[4]

Since $Q(\tau)$ is an autocorrelation function, its Fourier transform is a "power spectrum." Taking the transform of Equation (10) we have an example of the well-known result that the Fourier transform of a correlation integral is the product of the Fourier transform of the convolved members.

Of course, since $P(u)$ and $G(u)$ are both periodic in u, $Q(\tau)$ is also periodic with the time period $2\pi/\omega$. Thus the power spectrum is a line spectrum at frequencies which are multiples of $\omega/2\pi$.

The Fourier transform of Equation (10) is found without difficulty to be

$$W_Q(f) = \frac{1}{\omega} W_G^*(f/\omega)\, W_R(f/\omega)\, W_R^*(f/\omega) \quad ; \tag{12}$$

where

$$W_G(z) = \int_{-\infty}^{\infty} G(u) e^{-2\pi i z u}\, du \quad , \tag{13}$$

$$W_R(z) = \int_{-\infty}^{\infty} R(u) e^{-2\pi i z u}\, du \quad , \tag{14}$$

[4] If v had not been set equal to zero, the noise correlation function would not exist in this form, namely as a function of $u \equiv \theta' - \theta$. For the rectangular geometry noted in footnote 3, the noise correlation function exists as a function of $x' - x + v\tau$ where x' and x are coordinates which are rectangular analogues of θ' and θ, and $v\tau$ has the same meaning as above. Relations similar to Equation (10) and (11) also hold for this rectangular case.

and

$$W_Q(f) = \int_{-\infty}^{\infty} Q(\tau)\, e^{-2\pi i f \tau}\, d\tau \quad . \tag{15}$$

The symbol * denotes the complex conjugate.

Since $G(u)$ is an even function of u, $W_G^*(z) = W_G(z)$, and we may as well write

$$W_Q(f) = \frac{1}{\omega} W_G(f/\omega) \left| W_R(f/\omega) \right|^2 \quad . \tag{16}$$

Since $G(u)$, $R(u)$, and $Q(\tau)$ are periodic functions, the definitions of Equations (13), (14), and (15) need to be properly interpreted. Using a delta function notation, it is easy to show that

$$W_G(z) = \sum_{n=-\infty}^{\infty} \delta(z - n/2\pi)\, U_G(z) \quad , \tag{13a}$$

where

$$U_G(z) = 1/2\pi \int_0^{2\pi} G(u)\, e^{-2\pi i z u}\, du \quad , \tag{13b}$$

$$W_R(z) = \sum_{n=-\infty}^{\infty} \delta(z - n/2\pi)\, U_R(z) \quad , \tag{14a}$$

where

$$U_R(z) = 1/2\pi \int_0^{2\pi} R(u)\, e^{-2\pi i z u}\, du \quad , \tag{14b}$$

and

$$W_Q(f) = \sum_{n=-\infty}^{\infty} \delta(f - n\omega/2\pi)\, U_Q(f) \quad , \tag{15a}$$

where

$$U_Q(f) = \omega/2\pi \int_0^{2\pi/\omega} Q(\tau)\, e^{-2\pi i f \tau}\, d\tau \quad . \tag{15b}$$

Thus $W_Q(f)$ is a delta function or line spectrum. It is non-zero only for integral multiples of $\omega/2\pi$. The "line" character of the spectrum is due to the approximation v=0. If v is slightly different from zero, the lines are changed to narrow bands.

Equation (16) or the equivalent equation (10) exhibits the power spectrum of the output as a product of a "reticle spectrum" and a "noise spectrum" where the noise spectrum is the Fourier transform of G(u). We see then that once G(u) or its transform is computed, the output spectrum may be found either in terms of the convolution integral (10) or the spectral product (16). The "reticle spectrum" $W_R(z)$ or the "reticle correlation function" P(u), is easy to obtain in terms of the reticle geometry, and the definitions of Equations (13) to (15b). Thus, once G(u) is found, the output for various reticle functions is simple to compute. The problem of obtaining G(u), or its transform $W_G(z)$ by measurements or calculations, is therefore, of key importance. This problem we now examine.

DETERMINATION OF NOISE CORRELATION FUNCTION OR POWER SPECTRUM

From Equation (16) one sees that by measurements of $W_Q(f)$, the power transform of $Q(\tau)$, and measurements or calculations of $W_R(z)$, one can be division obtain $W_G(z)$, the transform of G(u). This, in principle is a valid method of obtaining $W_G(z)$, and indeed may turn out to be the most practical method. The reticle function $R(\theta)$ may be chosen for convenience. Reticle functions will generally be such that $W_R(z)$ is zero for certain values of z, so several reticle functions may have to be used to get enough information. One may also consider using a "delta function" reticle (single narrow transmitting slit) to pass "all" values of z. The results obtained for $W_G(z)$ will be valid only for the particular value of a used in the measurements, as reference to Equation (9a) shows. Thus one would have to repeat the process for each value of a considered.

Another, somewhat more fundamental approach is possible which we now wish to study.

Let us consider the noise correlation function G(u), as defined in Equation (9a),

$$G(u) = \int_0^a \int_0^a dr\, dr'\, rr' \psi \left[r^2 + r'^2 - 2rr' \cos u \right]^{1/2} \tag{9a}$$

From the form of this integral we see that G(u) is determined by the values of $\psi(d)$ for the range of d between 0 and 2a. (If v is not zero, reference to Equation (4a) or Figure 1 shows that values of ψ for a somewhat larger upper value of d are needed for the computation of $Q(\tau)$. This suggests that direct measurements $\psi(d)$ for such ranges (rather than measurements such as have been made up to the present, of the power spectrum) might be undertaken. For the purpose of obtaining, through expressions such as have been derived, insight into the output noise of the reticle, the correlation function is the quantity most directly needed. In principle one can Fourier-transform spectral measurements into correlation information, but in practice large errors may be introduced in attempting to transform the necessary incomplete measurements.

To proceed further on the theoretical side, we may try to find plausible forms for $\psi(d)$, which may be in agreement with the measurements so far, and see how these influence G(u) and its transform. As mentioned above, the limited amount of available experimental material bearing on this question has been expressed in terms not of ψ, but of its Fourier transform. Thus our comparisons at present between theoretical expressions and experiment can be made most conveniently in terms of the power spectra. The two-dimensional power spectrum $W_B(k_x, k_y)$ and the two-dimensional correlation function $\psi_B(x, y)$ are Fourier transforms of each other. That is,

$$W_B(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_B(x, y) \exp\left[-2\pi i(k_x x + k_y y)\right] dx\, dy\,, \tag{17}$$

and

$$\psi_B(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_B(k_x, k_y) \exp\left[2\pi i(k_x x + k_y y)\right] dk_x\, dk_y\,. \tag{18}$$

The writings and measurements of R. C. Jones and others, including measurements of sky background at The Ramo-Wooldridge Corporation, suggest the form

$$W_B(k_x, k_y) = M\left[k_x^2 + k_y^2\right]^{-3/2} , \tag{19}$$

where M is a constant, at least for fairly high wave numbers (number of waves per radian). More exactly, the measurements so far are not inconsistent with this form. However, this form is not acceptable for low wave numbers, because the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M\left[k_x^2 + k_y^2\right]^{-3/2} dk_x\, dk_y , \tag{20}$$

that is, the integral of the spectrum over all wave numbers, diverges. The infinite value is due to the pole at $k_x = k_y = 0$.

A way to adjust Equation (19) so as to eliminate this trouble and still not be inconsistent with the experimental data, has already been suggested in the writings of R. C. Jones. This is to use the form

$$W_B(k_x, k_y) = M\left[\alpha^2 + k_x^2 + k_y^2\right]^{-3/2} \tag{21}$$

If α^2 is not too large, this preserves agreement with the experimental results as out subsequent discussion will show.

First, let us obtain the correlation function associated with Equation (21); we must evaluate the Fourier integral,

$$\psi(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M\left[\alpha^2 + k_x^2 + k_y^2\right]^{-3/2} \exp\left[2\pi i(k_x x + k_y y)\right] dk_x\, dk_y . \tag{22}$$

This turns out to be

$$\psi(x, y) = \frac{\pi M}{\alpha} e^{-2\pi\alpha d} , \tag{23}$$

where

$$d = (x^2 + y^2)^{1/2} .$$

This result shows that $\psi(x, y)$ is a function only of $d = (x^2 + y^2)^{1/2}$ and thus indicates that the choice of the form (21) is consistant with our previous assumption of isotropic background. This property of the correlation function $\left[\text{dependence upon } (x^2 + y^2)^{1/2}\right]$ can be proven to hold for power spectra of the form

$$W(k_x, k_y) = \text{function of } (k_x^2 + k_y^2) . \tag{24}$$

The measurements of $W(k_x, k_y)$ that are known to the writer really measure the one-dimensional power spectrum — more precisely, the one-dimensional Fourier transform of the correlation function. For the form (23), this one-dimensional transform is

$$\begin{aligned} W_B(k) &= \int_{-\infty}^{\infty} \frac{\pi M}{|\alpha|} e^{-2\pi|\alpha|d} e^{-2\pi i dk} dd \\ &= \frac{M}{4\pi|\alpha|} \frac{1}{\alpha^2 + k^2} . \end{aligned} \tag{25}$$

The function $(\alpha^2 + k^2)^{-1}$, a normalized form of Equation (25), is plotted in Figure 2, versus k for various values of α. It is seen that asymptotically for high wave numbers, this spectrum follows a k^{-2} law. The value of α used determines how the curve deviates from the (-2) law. The several sets of measurements results that have been seen by the writer (schematically indicated by dashed line in Figure 2), are not inconsistent with this type of function, but it is difficult to make a

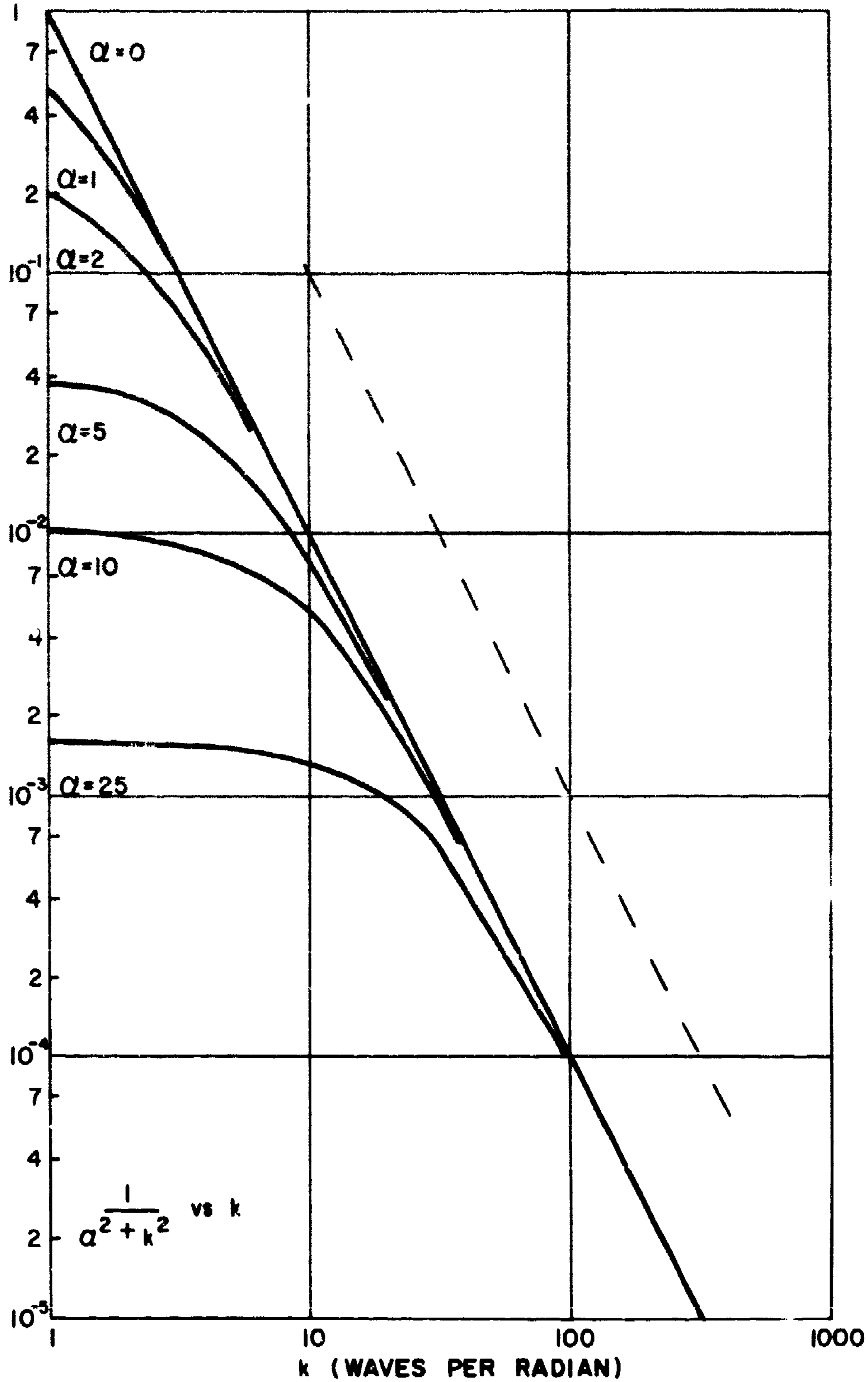

FIG. 2 THEORETICAL ONE-DIMENSIONAL POWER SPECTRUM. DASHED CURVE INDICATES APPROXIMATE SHAPE OF MEASURED POWER SPECTRA (LEVEL ARBITRARY).

judgement as to what the best value of α would be. Perhaps various background conditions (which give widely differing absolute levels of background) all give the same asymptotic (2) law, but vary in their effective values of α. If all that is known about the measured spectrum is, for example, that it follows a fairly accurate (-2) law in the range say above 10 waves per radian, then it is seen that any value of α up to about 5 is consistent with such measurements. This suggests that since the high k part of the spectrum has a form independent of α, some of the results of the subsequent analysis of output noise may also be independent of α. As measurements become more complete, it may be possible to choose more accurately between values of α or perhaps, to pick a more accurate form for the autocorrelation function. It is not unlikely that as measurements of background become more accurate, the (-2) law and the form $e^{-2\pi\alpha d}$ may have to be modified. Measurements aimed at directly measuring the background correlation function could perhaps settle this point more quickly than the spectral measurements. For exploratory purposes, however, it seems to be of some value to study G(u) and its Fourier transform for a range of values of α. At the time of writing, the function G(u) and its transform are being numerically studied.

For the time being, the Gaussian form $\exp(-p d^2)$ for the correlation function, which is often a likely candidate in similar problems has been eliminated from consideration because the range of the effective (-2) law, which seems to be an octave or more in the known measurements, cannot be fitted very well to the transform of $\exp(-p d^2)$ which is of the form $\exp(-q k^2)$; p and q are constants in the above; d and k have significance similar to that in Equations (22) and (25).

STUDY OF A PARTICULAR BACKGROUND CORRELATION FUNCTION

Studies and computations are under way of the properties of G(u) and its transform $W_G(z)$. At the time of writing these studies are incomplete, but a few preliminary remarks may be made.

Substituting the form (23) into the expression for G(u), we have

$$G(u) = \frac{\pi M}{\alpha} \int_0^a \int_0^a r r' \, e^{-2\pi\alpha d} \, dr \, dr' \quad . \tag{26}$$

Introducing the dimensionless parameters

$$\frac{r}{a} = x_1 \qquad \frac{r'}{a} = x_2 \qquad 2\pi\alpha a = h \,, \tag{27}$$

we obtain

$$F_h(u) = \frac{G(u)}{C} = a^4 \int_0^1 dx_1 \int_0^1 dx_2\, x_1 x_2 \exp\left[-h\left(x_1^2 + x_2^2 - 2x_1 x_2 \cos u\right)^{1/2}\right] \tag{28}$$

where

$$C = \frac{\pi M}{\alpha}$$

$F_h(u)$ is the significant function to be studied. The parameter h may be written as

$$h = \frac{a}{1/2\pi\alpha} \,.$$

Now $1/2\pi\alpha$ may be interpreted as a "correlation length," that is, $1/2\pi\alpha$ is the value of distance d for which the (normalized) correlation function $e^{-2\pi\alpha d}$ falls to one e^{th} of its peak value (which occurs for $d = 0$). The greater $1/2\pi\alpha$, the greater is the relative correlation of two points separated by a distance d. That is, the normalized correlation function falls more slowly from its peak value for high values of the correlation length $1/2\pi\alpha$. The parameter h is thus the ratio of the radius of the field of view and the correlation length of the background. The significant quantity affecting the properties of $F_h(u)$ is seen from Equation (28) to be this dimensionless parameter h. In addition, there is a scale factor of a^4, which simply means that all other things being equal, the output noise has a power level which is proportional to the square of the area of the field of view.

The dependence of $F_h(u)$ upon h and u is sketched in a purely qualitative way in Figure 3. The range $0 < u < 2\pi$ is shown; $F_h(u)$ is periodic in u (a result of the assumption $v = 0$). The behavior of the

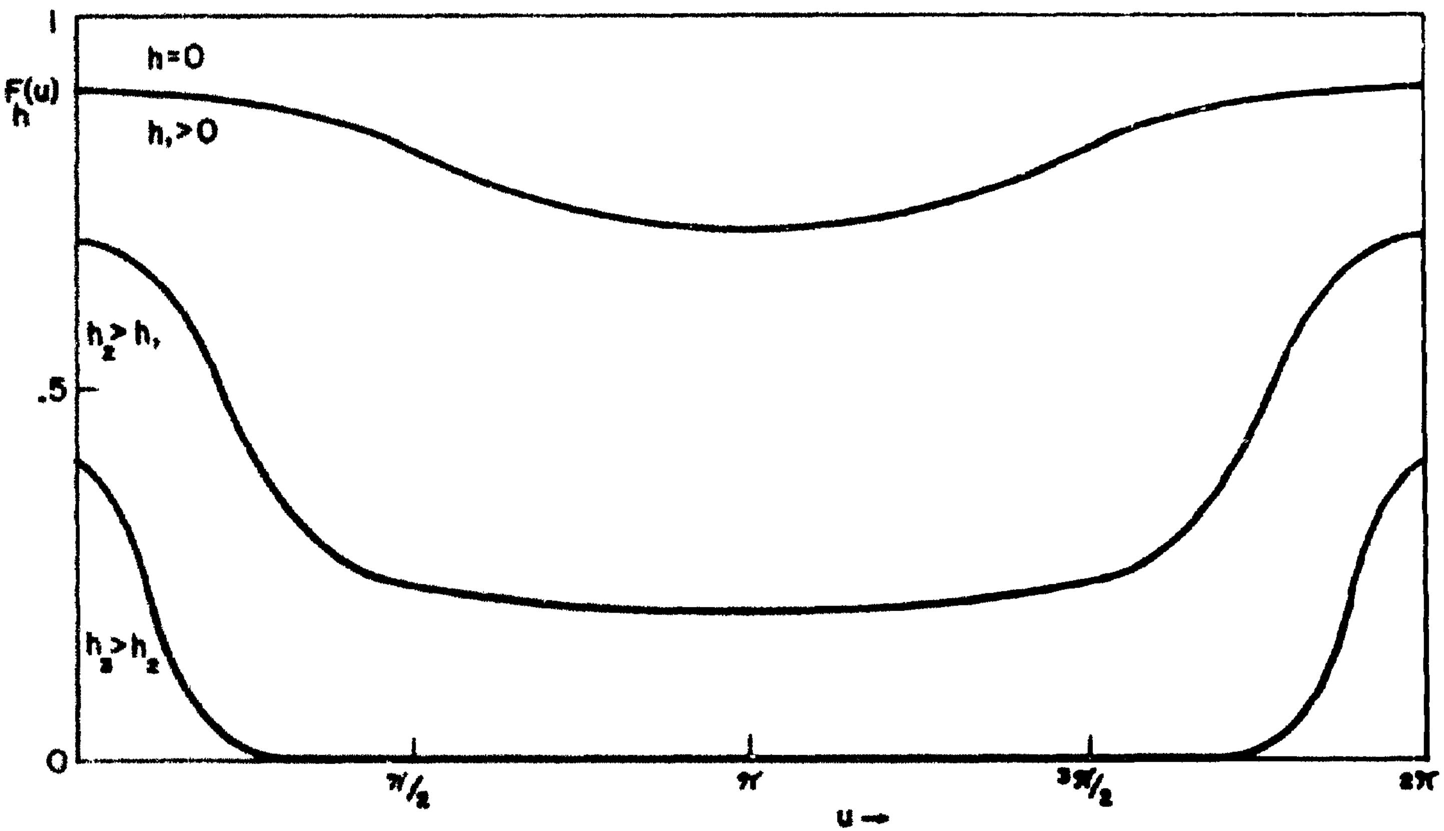

FIG. 3 $F_h(u)$ vs u – QUALITATIVE SKETCH

transform of $F_h(u)$ will be very roughly as follows: The higher values of h will yield a slower high frequency fall-off (more relative high frequency content); but the range and extent of these variations can only be obtained from the complete study.

To obtain a rough idea of what values of h may be of interest, let us imagine that the condition referred to above holds, namely that the measured spectrum follows roughly a (-2) power law in the range above 10 waves per radian; in the absence of further experimental details, any value of α up to about $\alpha = 5$, may provide a satisfactory fit. Let us take the radius a to be 2 degrees. Then $h_{max} = 2\pi\alpha a_{max} = 2\pi 5(2\pi/180) \approx 1.1$. The numerical work will cover values of h from 0 to 5.

Once $F_h(u)$ is computed for a sufficient range of h, it will serve the purpose of a "universal curve" applicable to any value of a and any reticle function $R(\theta)$. Its transform will likewise represent the power spectrum of the noise as seen by any reticle function. By multiplying this transform by the square of the transform of the reticle function, Equation (16), we may obtain the spectrum of the noise output of the reticle. This may then easily be combined with transforms of following stages.

This numerical work is only valid for the background correlation function $e^{-2\pi\alpha d}$. The method of attack, however, is still valid for any improved background functions that may later be obtained.

APPENDIX A

TRANSFORMATION OF A CERTAIN INTEGRAL EXPRESSION

This appendix concerns itself with the transformation of

$$I = \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \, G(\theta' - \theta) \, R(\theta) \, R(\theta' - \omega\tau) \, .$$

where $G(x) = G(x + 2\pi)$

and $R(x) = R(x + 2\pi)$

Introduce the change of variables

$$\left.\begin{aligned} u &= \theta' - \omega\tau - \theta \\ v &= \theta' - \omega\tau + \theta \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \theta &= \frac{u + v}{2} + \omega\tau \\ \theta &= \frac{v - u}{2} + \omega\tau \end{aligned}\right\}$$

The Jacobian of this transformation is

$$\begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = 1/2$$

We then have (see Figure A-1 for geometrical interpretation)

$$2I = \int_{2\pi - \omega\tau}^{-\omega\tau} du \, G(u + \omega\tau) \int_{v = 2\omega\tau - u}^{4\pi + u} dv \; R\left(\frac{v - u}{2}\right) R\left(\frac{v + u}{2}\right)$$

$$+ \int_{-\omega\tau}^{2\pi - \omega\tau} du \, G(u + \omega\tau) \int_{u}^{4\pi - 2\omega\tau - u} dv \; R\left(\frac{v - u}{2}\right) R\left(\frac{v + u}{2}\right) .$$

Introducing the change of variable $\frac{v - u}{2} = s$ we have

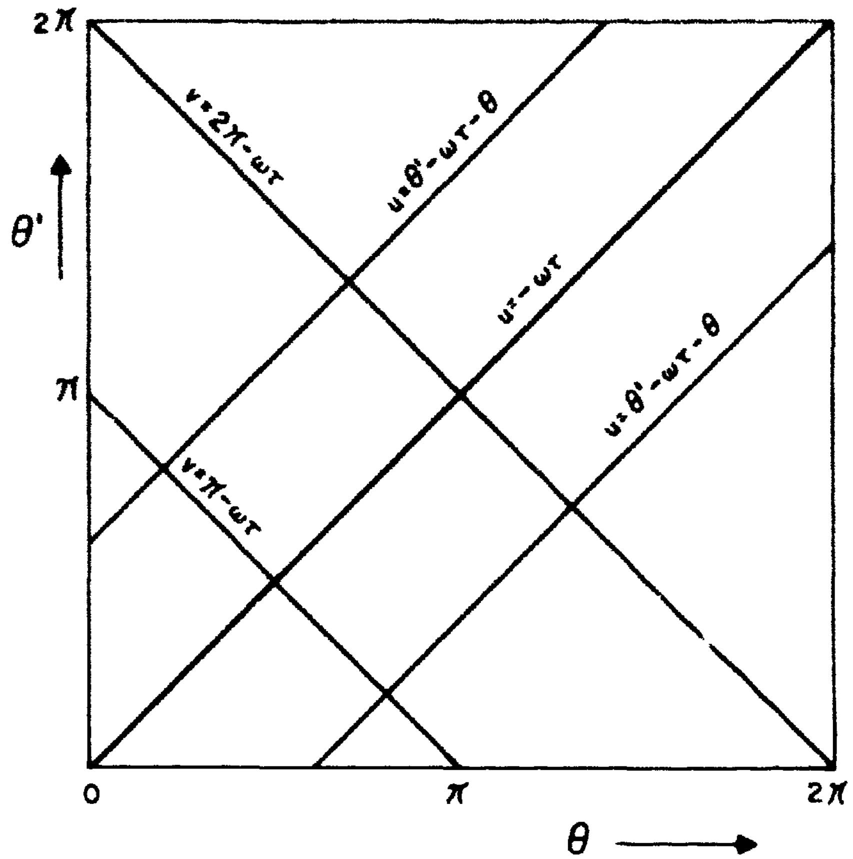

FIG. A-1

GEOMETRICAL INTERPRETATION OF CHANGE OF VARIABLE IN TEXT

$$\int dv\, R\left(\frac{v - u}{2}\right) R\left(\frac{v + u}{2}\right) = 2\int ds\, R(s)\, R(s + u)\ .$$

so that

$$I = \int_{-2\pi-\omega\tau}^{-\omega\tau} du\, G(u + \omega\tau) \int_{-\omega\tau-u}^{2\pi} ds\, R(s)\, R(s + u)$$

$$+ \int_{-\omega\tau}^{2\pi-\omega\tau} du\, G(u + \omega\tau) \int_{0}^{2\pi-\omega\tau-u} ds\, R(s)\, R(s + u)\ .$$

Now

$$\int_{-\omega\tau-u}^{2\pi} ds\, R(s)\, R(s + u) = \int_{-\omega\tau-u-2\pi}^{2\pi} ds\, R(s)\, R(s + u + 2\pi)$$

$$- \int_{-\omega\tau-u-2\pi}^{-\omega\tau-u} ds\, R(s)\, R(s + u + 2\pi)\ .$$

Calling

$$\int_{0}^{2\pi} ds\, R(s)\, R(s + u)\, ds = P(u) = P(u + 2\pi)\ ,$$

we have

$$\int_{-2\pi-\omega\tau}^{-\omega\tau} du\, G(u + \omega\tau) \left[\int_{-\omega\tau-u-2\pi}^{2\pi} ds\, R(s)\, R(s + u + 2\pi) - P(u + 2\pi)\right]$$

$$= \int_{-2\pi-\omega\tau}^{-\omega\tau} du\, G(u + 2\pi + \omega\tau) \left[\int_{-\omega\tau-u-2\pi}^{2\pi} ds\, R(s) R(s + u + 2\pi) - P(u + 2\pi)\right]$$

$$= \int_{-\omega\tau}^{-2\pi-\omega\tau} dv\, G(v + \omega\tau) \left[\int_{-\omega\tau-v}^{2\pi} ds\, R(s) R(s + v) - P(v)\right]$$

$$= \int_{-\omega\tau}^{-\omega\tau+2\pi} du\, G(u + \omega\tau) \left[\int_{-\omega\tau}^{2\pi} R(s) R(s + u)\, ds - P(u)\right]\ .$$

so that

$$I = \int_{-\omega\tau}^{-\omega\tau + 2\pi} du\, G(u + \omega\tau) \left[\int_{-\omega\tau - u}^{2\pi} R(s)R(s + u)\, ds \right.$$

$$\left. + \int_{0}^{2\pi - \omega\tau - u} R(s)R(s + u) ds - P(u) \right]$$

$$= \int_{-\omega\tau}^{-\omega\tau + 2\pi} du\, G(u + \omega\tau) \left[\int_{-2\pi}^{2\pi} R(s)R(s + u) ds - P(u) \right]$$

$$= \int_{-\omega\tau}^{-\omega\tau + 2\pi} du\, G(u = \omega\tau)\, P(u)$$

$$\boxed{= \int_{0}^{2\pi} du\, G(u + \omega\tau)\, P(u)}$$

where

$$P(u) = \int_{0}^{2\pi} R(s)\, R(s + u)\, ds\ .$$

APPENDIX B

ANALYSIS OF BACKGROUND MEASUREMENTS WITH CIRCULAR SCAN

In connection with the study of the effect of rotating reticle systems upon background, there is at present an interest in a program of background "gradient" measurements in which the background is scanned with a small field of view moving repetitively around a circle of radius a. The Fourier coefficients of the periodic output are taken with the object of relating these Fourier coefficients to the properties of the background. Operating under the assumption that the background is a stationary random function, we give an indication of the kind of reductions needed to convert such data into a background power spectrum or correlation function.

EXPRESSIONS FOR GENERAL PATHS

Before specializing to the case of the circular path we present some expressions which are valid for general paths.

Let us assume that our instrument measures the radiance B(s) [1] as a function of the running angular coordinate s along the path, and that this type of scan is repeated over many similar segments of the field, and that for each measurement along the path we obtain a set of Fourier coefficients.

A typical complex Fourier coefficient is

$$C(n) = \frac{1}{A} \int_0^A B(s)\ e^{-i2\pi(n/A)s}\ ds \tag{1}$$

where A is the length of the path.

---

[1] See footnote (1) of the main text.

The square of the magnitude of C(n) is

$$C(n)\, C^*(n) = \frac{1}{A^2} \int_0^A \int_0^A e^{-i2\pi(n/A)(s-p)}\, B(s)\, B(p)\; ds\, dp \qquad (2)$$

where n is any integer between $-\infty$ and ∞.

We shall assume that the measurements give us the $C(n)\, C^*(n)$.

To obtain average data we average over many paths or cases. We have

$$D_n \equiv \left\langle C(n)\, C^*(n) \right\rangle = \frac{1}{A^2} \int_0^A \int_0^A e^{-i2\pi(n/A)(s-p)} \left\langle B(s)\, B(p) \right\rangle ds\, dp \qquad (3)$$

where $\langle \; \rangle$ denotes the average over many cases (stochastic or ensemble average).

RESTRICTION TO ISOTROPIC BACKGROUND AND CIRCULAR OR STRAIGHT-LINE PATHS

Further evaluation of Equation (3) depends on the assumptions that are made on the form of $\left\langle B(s)\, B(p) \right\rangle$. Let us assume for the moment that

$$\left\langle B(s)\, B(p) \right\rangle = \phi(s - p) = \phi(p - s) \qquad (4)$$

i.e., that the average value of B(s) B(p) depends only upon the difference between the running coordinates s and p. It will be seen below that this assumption, which enables us to make progress in the interpretation of Equation (3), implies certain restrictions on the background and on the paths.

With this assumption, Equation (3) becomes

$$D_n = \frac{1}{A^2} \int_0^A \int_0^A e^{-i2\pi(n/A)(s-p)} \phi(s - p) \, ds \, dp \tag{5}$$

If now we change to the variables

$$\begin{aligned} u &= s - p \\ v &= s + p, \end{aligned} \tag{6}$$

we obtain, noting that the Jacobian of this transformation is 1/2,

$$D_n = \frac{1}{2A^2} \int_{-A}^{A} du \int_{|u|}^{2A-|u|} dv \, e^{-i2\pi(n/A)\,u} \phi(u) \tag{7}$$

$$= \frac{1}{2A^2} \int_{-A}^{A} (2A - 2|u|) \, e^{-i2\pi(n/A)\,u} \phi(u) \, du$$

$$= \frac{1}{A} \int_{-A}^{A} \left(1 - \frac{|u|}{A}\right) \cos \frac{2\pi nu}{A} \phi(u) \, du \tag{8}$$

Thus with the help of the assumption (4) we have reduced (3) from a double integral to a single integral. The interpretation of this integral will be discussed below; in particular we shall examine the extent to which information about the background random function can be extracted from (8). First however, we should pause to consider under which conditions $\langle B(s) \, B(p) \rangle$ will be of the form $\phi(s - p)$, so that the form (8) will be valid. Let us assume that the two-dimensional noise is isotropic; this means, as is discussed above, that the correlation function, which is with full generality a function of two space variables,

say the orthogonal variables x and y, is in this case a function only of $d = \left[x^2 + y^2\right]^{1/2}$. In other words $\langle B(s)B(p)\rangle$, which is a correlation function, is a function $\phi(d)$ of the scalar distance between the path points labelled s and p. However, this is still not sufficient for the validity of (4), in which $\langle B(s)B(p)\rangle$ is set equal to a function of $s - p$. In order for this to be valid, the distance between the points s and p should depend only upon the quantity $s - p$, and not upon s; this condition seems to be satisfied only if the scanning path is a straight line or a part of a circle. For an ellipse for example, the same path length difference $(s - p)$ will give different distances between s and p, depending upon what part of the elliptical path s and p refer to.

Thus, formula (8) refers to the case of isotropic noise and straight line or circular scans.

Equation (8) exhibits D_n as the cosine transform of the quantity $(1 - u/A)\ \phi(u)$. A possible procedure then, is to invert (8) to obtain $(1 - u/A)\ \phi(u)$, divide by $1 - u/A$ to find $\phi(u)$ and then from $\phi(u)$ find $\Psi(d)$, the correlation function of the background. If the path s has been a segment of a straight line, then u is the same as d and $\phi(u) = \phi(d) = \Psi(d)$. If the path s is a circular path (or part of a circular path), then the relation between u and d is that d is the chord connecting the ends of the circular arc whose length is u. That is, if the radius of the circle is a, $(d/2a) = \sin^{-1}(u/2a)$. We have then

$$\Psi(d) = \phi\left[2a \sin(d/2a)\right] \tag{9}$$

so that by a straightforward nonlinear scale change we can obtain $\Psi(d)$ having once obtained $\phi(u)$. We may then use $\Psi(d)$ to obtain G(u) the noise correlation function by means of Equation (9a) of the main body of this paper.

The main task then, is the inversion of Equation (8) to obtain $(1 - u/A)\ \phi(u)$. We have noted that the D_n is expressed as a cosine transform of $(1 - u/A)\ \phi(u)$. More precisely, however, the cosine transform of $(1 - |u|/A)\ \phi(u)$ that fully represents the function in the range o to A is

$$E_n = \frac{1}{A} \int_{-A}^{A} \left(1 - \frac{|u|}{A}\right) \phi(u) \cos \frac{2\pi nu}{2A} \, du \tag{10}$$

Comparing Equation (10) with Equation (8) we see that the D_n are not quite identical with the E_n but that $E_n = D_{n/2}$. Since however the D_n are available only for integral n, half the E_n are missing. To obtain all the E_n we must interpolate between the values of D_n; that is for example on the curve of D_n versus n, E_3 is "$D_{3/2}$".

Once having obtained a complete set of E_n, half of which are the D_n and the others interpolated values, we may find $(1 - u/A)\ \phi(u)$ by the inversion formula

$$\left(1 - \frac{u}{A}\right) \phi(u) = \frac{E_o}{2} + \sum_{n=1}^{\infty} E_n \cos \frac{\pi nu}{A} \tag{11}$$

The task of interpolation between the D_n may be rendered somewhat more hopeful by the fact that $(1 - u/A)\phi(u)$ should be a correlation function so that none of the E_n should be negative. Having obtained $(1 - u/A)\ \phi(u)$ the rest of the procedure outlined above for obtaining $\Psi(d)$ may be followed. If desired one may then obtain the one-dimensional power spectrum of the background by Fourier-transforming $\Psi(d)$.

It is seen from the above discussion that the procedure of obtaining $\Psi(d)$ from the D_n would be subject to considerable error. This suggests that if our aim is to compute G(u), in which $\Psi(d)$ itself appears, then the measurements of background be arranged to directly obtain $\Psi(d)$ rather than data like the D_n.

Thus for example, one could measure the average value over s of $B(s)\ B(s + u)$ where s and s + u are, say, positions in the circular scan path. This average value, itself averaged over a sufficiently large number of cases, is then the function $\phi(u)$ discussed above.

$\Psi(d)$ is then easily obtained. Straight line paths could, of course, also be used, yielding $\Psi(d)$ directly.

A further remark needs to be made concerning the D_n. It can be shown[2] that the integral

$$\int_{-A}^{A} \left(1 - \frac{|u|}{A}\right) \phi(u) \cos 2\pi f u \; du$$

which appears in Equation (8), approaches $\int_{-\infty}^{\infty} \phi(u) \cos 2\pi f u \, du$ as A approaches infinity. This last integral, however, is the power spectrum associated with the correlation function $\phi(u)$. As A approaches infinity, moreover, $\phi(u)$ approaches $\Psi(d)$. Therefore for large A the D_n are approximately equal to $(1/A)\, W_B\,(n/A)$, where the one dimensional power spectrum $W_B\,(k)$ is defined, as in Equation (25) of the main text, as the Fourier transform of $\Psi(d)$:

$$W_B\,(k) = \int_{-\infty}^{\infty} \Psi(d)\, e^{-2\pi i d k} \; dd$$

$$= \int_{-\infty}^{\infty} \Psi(d) \cos 2\pi d k \;\; dd \, .$$

For a given value of A, the statement is more nearly true, on the average, for high n.

[2] Titchmarsh. "Theory of Fourier Integrals." p. 36.